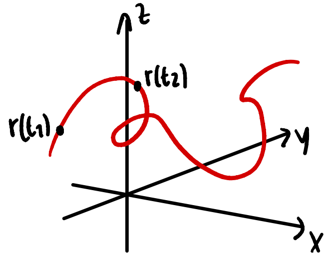
Krivulje in ploskve

Ali je krivulja množica točk v prostoru?

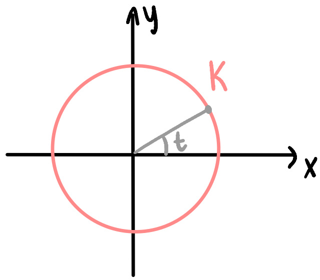


Def: Pot je preslikava $\vec{r}: I \to \mathbb{R}^3$; $I=[a,b]$, $t \in I$:
$\vec{r}(t) \in \mathbb{R}^3$, $\vec{r}(t) = (x(t), y(t), z(t))$
Pot je <u>preslikava</u>, krivulja pa je <u>slika poti $\vec{r}$</u>

Pravimo tudi, da je pot $\vec{r}$ parametrizacija krivulje.

Primer: Parametriziraj krivuljo $K = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$

$\vec{r}: I \to \mathbb{R}^2$ in $\vec{r} = K$
$t \to \vec{r}(t)$



Več možnih parametrizacij

$\vec{r}_1: [0, 2\pi] \to \mathbb{R}^2$
$t \to (\cos t, \sin t)$
Pot $\vec{r}_1$ je parametrizacija K

$\vec{r}_2: \mathbb{R} \to \mathbb{R}^2$
$\vec{r}_2(t) = (\cos t, \sin t)$
Pot $\vec{r}_2$ je parametrizacija, a preslikava ni injektivna

$\vec{r}_3: [0, 2\pi) \to \mathbb{R}^3$
$\vec{r}_3(t) = (\cos t, \sin t)$
Pot $\vec{r}_3$ je parametrizacija in je injektivna.

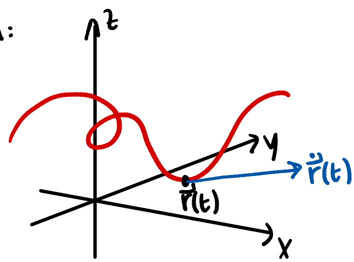
$\vec{r}_4 = (\cos(t^2), \sin(t^2))$ $t \in \mathbb{R}$ $\Rightarrow$ $\text{im}\,\vec{r}_4 = K$

$\vec{r}_5 = (\cos(\ln t), \sin(\ln t))$ $t \in (0,\infty)$ $\Rightarrow$ $\text{im}\,\vec{r}_5 = K$

$\vdots$

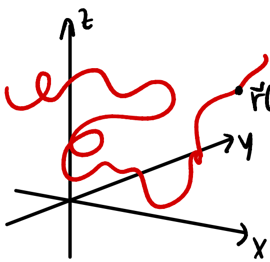
Odvod: $\vec{r}(t) = (x(t), y(t), z(t)) \Rightarrow \dot{\vec{r}}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$

→ Geometrijski pomen odvoda:



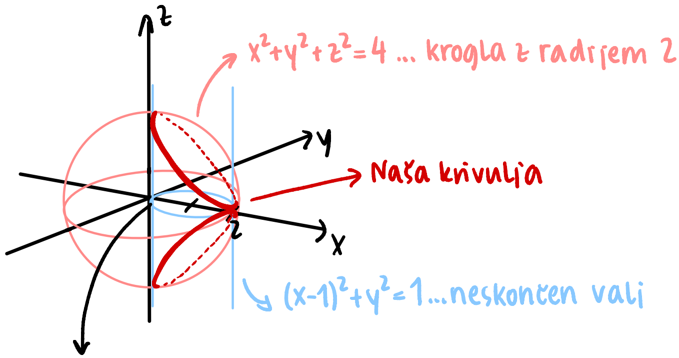
$\dot{\vec{r}}(t)$ je vektor, ki je tangenten na krivuljo v točki $\vec{r}(t)$.

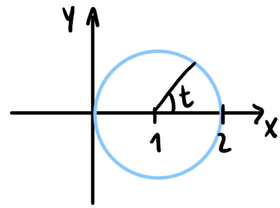
→ Fizikalni pomen:



$\vec{r}(t)$ položaj muhe ob času t
$\Downarrow$
$\dot{\vec{r}}(t)$ je hitrost muhe ob času t

(1) Parametriziraj krivuljo dano z $x^2+y^2+z^2=4$ in $(x-1)^2+y^2=1$ in pri $x=1, y<0, z>0$ zapiši enačbo tangente na krivuljo.





Najprej parametrizirajmo modri krog: $x=1+\cos t$, $y=\sin t$ $(t\in\mathbb{R})$

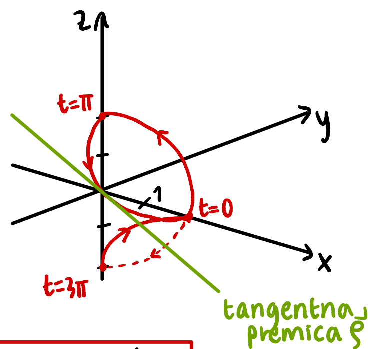
Zdaj mora biti z tak, da ustreza pogoju za kroglo:

$$x^2+y^2+z^2=4$$
$$(1+\cos t)^2+\sin^2 t+z^2=4$$
$$z^2=4-\sin^2 t-1-2\cos t-\cos^2 t = 3-1-2\cos t = 2\cdot(1-\cos t)=$$
$$=2\cdot\left(\sin^2\tfrac{t}{2}+\cancel{\cos^2\tfrac{t}{2}}-\cancel{\cos^2\tfrac{t}{2}}+\sin^2\tfrac{t}{2}\right)$$
$$\Rightarrow z^2=4\cdot\sin^2\tfrac{t}{2} \Rightarrow \underline{\underline{z=\pm 2\cdot\sin\tfrac{t}{2}}}$$

$$\boxed{\text{Parametrizacija } \vec{r}(t)=\left(1+\cos t, \sin t, 2\cdot\sin\tfrac{t}{2}\right)}$$

$$\left.\begin{array}{l} t=0\Rightarrow(2,0,0)\\ t=\pi\Rightarrow(0,0,2)\\ t=2\pi\Rightarrow(2,0,0)\end{array}\right] t\in[0,2\pi]$$

$$\left.\begin{array}{l} t=3\pi\Rightarrow(0,0,-2)\\ t=4\pi\Rightarrow(2,0,0)\end{array}\right] t\in[2\pi,4\pi] \quad \sin t\le 0$$



$$\boxed{t\in[0,4\pi] \quad \operatorname{im}\vec{r}=K}$$

Tangenta na krivuljo $x=1\Rightarrow 1+\cos t=1 \Leftrightarrow \cos t=0$ $t\in\{\frac{\pi}{2},\frac{3\pi}{2},\dots\}$
zanima nas tangenta pri $t=3\pi/2$ (ker $z>0$ in $y<0$)

$$\vec{r}\left(\tfrac{3\pi}{2}\right)=(1,-1,\sqrt{2})$$

$$\dot{\vec{r}}(t)=\left(-\sin t,\cos t,\cos\tfrac{t}{2}\right)\Rightarrow \dot{\vec{r}}\left(\tfrac{3\pi}{2}\right)=\left(1,0,-\tfrac{\sqrt{2}}{2}\right)$$

Imamo tangentno premico $\vec{\rho}=(1,-1,\sqrt{2})+s\left(1,0,-\tfrac{\sqrt{2}}{2}\right)$

26.11.2020

DOLŽINA POTI (KRIVULJE)

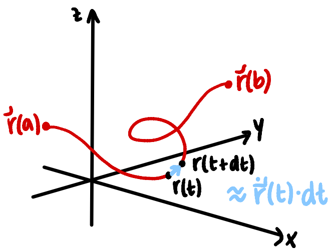
$\vec{r}:[a,b]\to\mathbb{R}^3$

Taylorjev razvoj $dt\approx 0$

$\vec{r}(t+dt)-\vec{r}(t) = \cancel{r(t)}+\dot{\vec{r}}(t)dt-\cancel{\vec{r}(t)}$

Dolžina tega koščka je torej $|\dot{\vec{r}}|dt$
To potem naredimo za vse koščke:

$$L=\int_a^b|\dot{\vec{r}}(t)|dt$$



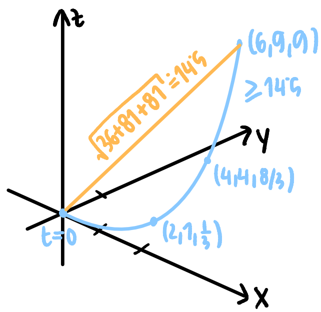
Opomba: Ali se to ujema z dolžino krivulje? $\Rightarrow$ Če $\vec{r}:[a,b]\to\mathbb{R}^3$ injektivna, potem se $\int_a^b|\dot{\vec{r}}(t)|dt$ ujema z dolžino krivulje (oz. $|\dot{\vec{r}}|\neq 0$ povsod)

$\vec{r}(t)=(\cos t,\sin t)$, $t\in[0,4\pi]$ → To sta 2 kroga po enotski krožnici

$\dot{\vec{r}}(t)=(-\sin t,\cos t)\Rightarrow|\dot{\vec{r}}|=1$

$\rightarrow L=\int_0^{4\pi}1\cdot dt=4\pi=2\cdot$ obseg krožnice

(1) Dana je krivulja K parametrizirana z $\vec{r}(t)=(2t,t^2,\frac{1}{3}t^3)$ za $t\in[0,3]$. Določi dolžino krivulje K.



$\dot{\vec{r}}(t)=(2,2t,t^2)\Rightarrow|\dot{\vec{r}}|=\sqrt{4+4t^2+t^4}=\sqrt{(2+t^2)^2}=(2+t^2)$

$$L=\int_0^3|\dot{\vec{r}}(t)|dt=\int_0^3(2+t^2)dt=2t+\frac{t^3}{3}\Big|_0^3=6+9=15$$

NARAVNA PARAMETRIZACIJA

$\vec{r}:[a,b]\to\mathbb{R}^3$
$t\to\vec{r}(t)$

Pravimo, da je t naravni parameter, če je $|\dot{\vec{r}}(t)|\equiv 1$ (Tipično namesto t potem pišemo s)

$\vec{r}(s):[0,a]\to\mathbb{R}^3$, $|\vec{r}\,'(s)|\equiv 1$

Dolžina krivulje $s\in[0,a]$: $L(a)=\int_0^a 1\cdot ds=a$

Naravni parameter je točno tisti parameter, ki meri dolžino krivulje

Tipično $\vec{r}: I \to \mathbb{R}^3$, $|\dot{\vec{r}}(t)| \neq 1$ hočemo naravno reparametrizirati (to se vedno da, če je $\dot{\vec{r}}(t) \neq 0$ povsod)

$$s := \int_a^t |\dot{\vec{r}}(\gamma)| d\gamma \Rightarrow s(t)$$

inverz

odvajamo po t

$$\dot{s} = |\dot{\vec{r}}(t)| \neq 0 \qquad t(s) \quad \left(t'(s) = \frac{1}{|\dot{\vec{r}}(t)|}\right)$$ → odvod inverzne funkcije

Reparametrizacija krivulje: $\vec{\rho}(s) = \vec{r}(t(s))$. Ali je s naravni parameter?

$$\vec{\rho}'(s) = \dot{\vec{r}} \cdot t' = \dot{\vec{r}} \cdot \frac{1}{|\dot{\vec{r}}|} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} = \text{enotski vektor} \Rightarrow |\vec{\rho}'| = 1$$

Primer: $\vec{r}(t) = (3\cdot\cos t, 3\cdot\sin t)$ $t \in [0, 2\pi]$

↳ Krožnica z radijem 3

$\dot{\vec{r}} = (-3\cdot\sin t, 3\cdot\cos t)$, $|\dot{\vec{r}}| = 3 \neq 1$ ⇒ t ni naravni parameter

$$s = \int_0^t 3 d\gamma = 3t \Rightarrow t = \tfrac{1}{3}s$$

$\vec{\rho}(s) = (3\cdot\cos\frac{s}{3}, 3\cdot\sin\frac{s}{3})$ Naravna parametrizacija krožnice radija 3

$s \in [0, 6\pi]$

Opomba: Vzamemo kakšno drugo spodnjo mejo: $s = \int_{\pi/2}^t 3 d\gamma = 3(t - \frac{\pi}{2}) \Rightarrow t - \frac{\pi}{2} = \frac{1}{3}s$

$\vec{\rho}(s) = (3\cdot\cos(\frac{\pi}{2} + \frac{s}{3}), 3\cdot\sin(\frac{\pi}{2} + \frac{s}{3}))$, $s \in [-\frac{3\pi}{2}, \frac{9\pi}{2}]$ $\qquad t = \frac{1}{3}s + \frac{\pi}{2}$

To je spet naravna parametrizacija, samo krivuljo drugje začnemo

(2) Dana naj bo parametrizacija $\vec{r}(t) = (2t, t^2, \frac{1}{3}t^3)$. Izračunaj w'', kjer je $w = x^2 + 4y^2 + 9z^2$ in s črtico je označen odvod po naravnem parametru.

Poiščimo najprej naravno reparametrizacijo dane parametrizacije

$\vec{r}(t) = (2t, t^2, \frac{1}{3}t^3)$, $\dot{\vec{r}}(t) = (2, 2t, t^2)$, $|\dot{\vec{r}}| = 2 + t^2$ (od prej) $\neq 1$ ⇒ t ni naravni param.

$$s = \int_0^t |\dot{\vec{r}}(\gamma)| d\gamma = \int_0^t (2 + \gamma^2) d\gamma = 2\gamma + \tfrac{1}{3}\gamma^3 \Big|_0^t = 2t + t^3/3$$ → Nalogo bomo poskusili rešiti, ne da bi izrazili $t(s)$

Mathematica: $t = \dfrac{2\cdot\sqrt[3]{2}}{\sqrt[3]{\sqrt{32 + 9s^2} - 3s}} - \dfrac{\sqrt[3]{\sqrt{32 + 9s^2} - 3s}}{\sqrt[3]{2}}$ → Dostikrat se inverza sploh ne da izraziti z element. funkcijami

$\Rightarrow \vec\rho(s) = (\underbrace{2\cdot t(s)}_{x(s)}, \underbrace{t(s)^2}_{y(s)}, \underbrace{t(s)^3/3}_{z(s)})$, $W(s) = x(s)^2 + 4y(s)^2 + 9z(s)^2$

$W''(s) = \dots$

Naporno!

Ali lahko izračunamo W'', brez da bi eksplicitno poznali $t(s)$?

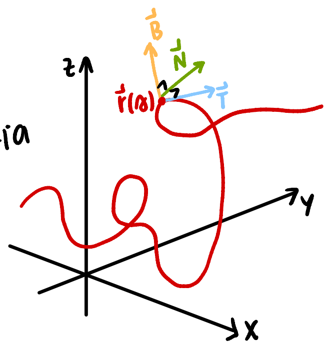
$$s = 2t + \tfrac{1}{3}t^3 \Rightarrow \dot s = 2 + t^2$$
$$\Downarrow$$
$$t = t(s) \to \boxed{t' = \frac{1}{2+t^2}}$$

Ker je to odvod inverzne funkcije

Denimo, da imamo neko količino $\varphi(t) \to$ izrazimo preko naravnega parametra s:

$$\Phi(s) = \varphi(t(s))$$
$$\Phi' = \dot\varphi \cdot t' = \frac{\dot\varphi(t)}{2+t^2}$$
$$\Phi'' = \left(\frac{\dot\varphi}{2+t^2}\right)^{\cdot} \cdot t' = \left(\frac{\dot\varphi}{2+t^2}\right)^{\cdot} \cdot \frac{1}{2+t^2}$$
$$W = x^2 + 4y^2 + 9z^2 = 4t^2 + 4t^4 + t^6$$
$$W(s) = 4t^2(s) + 4t(s)^4 + t(s)^6$$
$$W' = (8t + 16t^3 + 6t^5)\cdot t' = \frac{8t + 16t^3 + 6t^5}{2+t^2}$$
$$W'' = \left(\frac{8t+16t^3+6t^5}{2+t^2}\right)^{\cdot} \cdot t' = \frac{(8+48t^2+30t^4)(2+t^2) - 2t(8t+16t^3+6t^5)}{(2+t^2)^2} \cdot \frac{1}{2+t^2} =$$
$$= \frac{16+96t^2+60t^4+8t^2+48t^4+30t^6-16t^2-32t^4-12t^6}{(2+t^2)^3} = \boxed{\frac{16+88t^2+76t^4+18t^6}{(2+t^2)^3} = W''}$$

FRENETOVA BAZA



$\vec r(s)$ ... naravno parametrizirana krivulja

$\vec T = \vec r'$, $\vec N = \frac{\vec r''}{|\vec r''|}$ $(\vec r'' \neq 0)$, $\vec B = \vec T \times \vec N$

Fleksijska ukrivljenost $\varkappa = |\vec r''|$

Torzijska ukrivljenost $\tau = \frac{\vec r' \cdot (\vec r \times \vec r'')}{|\vec r''|^2}$

Povzetek: za $\vec r(s)$ $(\varkappa \neq 0)$ imamo 3 vektorje $\vec T = \vec r'$, $\vec N = \frac{\vec T'}{|\vec T'|}$, $\vec B = \vec T \times \vec N$, ki so enotski in pravokotni drug na drugega v vsaki točki. $\vec T, \vec N, \vec B$ ... Frenetova baza.

Frenetove formule:

$\vec T' = \varkappa \vec N$, $\vec N' = \tau \vec B - \varkappa \vec T$, $\vec B' = -\tau \vec N$

Obstaja izrek: Če imamo funkciji $f(s)>0,\ g(s) \Rightarrow$ vedno $\exists!\ \vec{r}(s)$ tako da je
$\varkappa(s)=f(s)$ in $\tau(s)=g(s)$

do rotacij & translacij v prostoru

Kaj pa če $\vec{r}(t)$ ni naravna parametrizacija? $\to \vec{T},\vec{N},\vec{B},\varkappa,\tau$?

$$\vec{T}=\frac{\dot{\vec{r}}}{|\dot{\vec{r}}|},\quad \vec{B}=\frac{\dot{\vec{r}}\times\ddot{\vec{r}}}{|\dot{\vec{r}}\times\ddot{\vec{r}}|},\quad \vec{N}=\vec{B}\times\vec{T},\quad \varkappa=\frac{|\dot{\vec{r}}\times\ddot{\vec{r}}|}{|\dot{\vec{r}}|^3},\quad \tau=\frac{\dot{\vec{r}}(\ddot{\vec{r}}\times\dddot{\vec{r}})}{|\dot{\vec{r}}\times\ddot{\vec{r}}|^2},\quad R=\frac{1}{\varkappa}\ \dots \text{Radij ukrivljenosti}$$

(3) Dana je krivulja s parametrizacijo $\vec{r}(t)=\left(\frac{1}{4}t^4,\frac{1}{3}t^3,\frac{1}{2}t^2\right)$; $t\in\mathbb{R}$

recimo $t\geq 0$, da bo lažje koreniti

a) Določi $\vec{T},\vec{N},\vec{B},\varkappa,\tau$ za $\forall t\in\mathbb{R}$

b) Določi enačbo binormalne in pritisnjene ravnine v točki $t=1$

a) $\vec{r}(t)=\left(\frac{1}{4}t^4,\frac{1}{3}t^3,\frac{1}{2}t^2\right)$

$\dot{\vec{r}}(t)=(t^3,t^2,t)$ $\qquad \dot{\vec{r}}\times\ddot{\vec{r}}=(-t^2,2t^3,-t^4)$ $\qquad |\dot{\vec{r}}|=\sqrt{t^2+t^4+t^6}=t\sqrt{1+t^2+t^4}$

$\ddot{\vec{r}}(t)=(3t^2,2t,1)$ $\qquad \ddot{\vec{r}}\times\dddot{\vec{r}}=(-2,6t,-6t^2)$

$\dddot{\vec{r}}(t)=(6t,2,0)$

$$\vec{T}=\frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}=\frac{(t^3,t^2,t)}{t\sqrt{t^4+t^2+1}}=\frac{(t^2,t,1)}{\sqrt{t^4+t^2+1}},\qquad \vec{B}=\frac{(-t^2,2t^3,-t^4)}{\sqrt{t^4+4t^6+t^8}}=\frac{(-1,2t,-t^2)}{\sqrt{1+4t^2+t^4}}$$

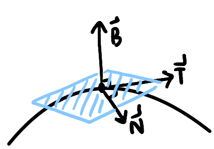
$$\vec{N}=\vec{B}\times\vec{T}=\dots=\frac{(t^3+2t,1-t^4,-2t^3-t)}{\sqrt{t^8+5t^6+6t^4+5t^2+1}}$$

$$\varkappa=\frac{|\dot{\vec{r}}\times\ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}=\frac{\sqrt{t^4+4t^6+t^8}}{t^3\sqrt{t^4+t^2+1}^3}=\frac{\cancel{t^2}\sqrt{1+4t^2+t^4}}{t^{\cancel{3}}\sqrt{1+t^2+t^4}^3}$$

$$\tau=\frac{\dot{\vec{r}}(\ddot{\vec{r}}\times\dddot{\vec{r}})}{|\dot{\vec{r}}\times\ddot{\vec{r}}|^2}=\frac{(t^3,t^2,t)\cdot(-2,6t,-6t^2)}{t^4+4t^6+t^8}=\frac{-2t^3+\cancel{6t^3}-\cancel{6t^3}}{t^4(1+4t^2+t^4)}=\frac{-2}{t(1+t^2+t^4)}$$

b) Pritisnjena ravnina $=\perp\vec{B}$

Binormalna ravnina $=\perp\vec{N}$

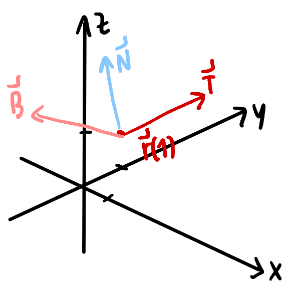


Poglejmo vse količine v točki $t=1$:

$\vec{r}(1)=\left(\frac{1}{4},\frac{1}{3},\frac{1}{2}\right)$

$\vec{T}(1)=\frac{(1,1,1)}{\sqrt{3}}$; $B(1)=\frac{(-1,2,-1)}{\sqrt{6}}$;

$\vec{N}(1)=\vec{B}(1)\times\vec{T}(1)=\frac{1}{\sqrt{18}}\cdot(3,0,-3)=\frac{1}{\sqrt{2}}(1,0,-1)$



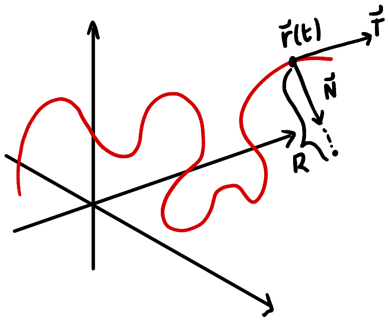
Pritisnjena ravnina gre skozi točko $\left(\frac{1}{4},\frac{1}{3},\frac{1}{2}\right)$ in je $\perp$ na $\frac{(-1,2,-1)}{\cancel{\sqrt{6}}}$ ← normala

$$\Rightarrow -x+2y-z=-\frac{1}{4}+\frac{2}{3}-\frac{1}{2}=-\frac{1}{12}$$

$$x-2y+z=\frac{1}{12}$$

Binormalna ravnina gre skozi točko $\left(\frac{1}{4},\frac{1}{3},\frac{1}{2}\right)$ in je $\perp$ na $(1,0,-1)$: $-x+z=\frac{1}{4}$

Naloga: Dana je krivulja $\vec{r}: I \to \mathbb{R}^3$, $t \to \mathbb{R}^3$. Za vsak $t \in I$ poišči središče pritisnjene kr.



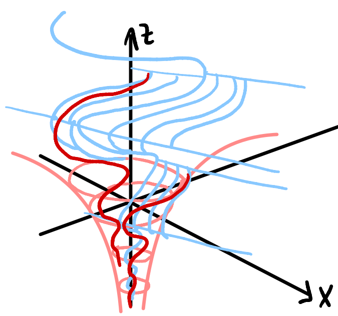
$S(t)=?$

$$S(t)=\vec{r}(t)+\frac{1}{\varkappa(t)}\cdot\vec{N}=\vec{r}(t)+R\cdot\vec{N}$$

za radij ukrivljenosti se premaknemo v smeri vektorja normale

(4) Parametriziraj krivuljo dano s pogoji $y=e^z\cdot\sin z$, $x^2+y^2=e^{2z}$ ter pri $z=0$ $(x>0)$ določi fleksijsko & torzijsko ukrivljenost.

Lahko gledamo kot funkcijo v yz-ravnini, raztegnjeno v smeri osi x



$z=0$: $x^2+y^2=1$

$z=1$: $x^2+y^2=e^2 \doteq 9$

$\vdots$

Pri vsakem z je to krožnica z radijem e^z

K leži na takšni ploskvi in je torej presek

$$x^2+y^2=e^{2z} \cap y=e^z\cdot\sin z$$

Parametrizirajmo K: $t=z$

$$\vec{r}(t)=(x(t),\ e^t\cdot\sin t,\ t)$$

$$x(t)^2+e^{2t}\cdot\sin^2 t=e^{2t}$$

$$x(t)^2=e^{2t}(1-\sin^2 t)=e^{2t}\cdot\cos^2 t$$

$\Rightarrow x(t)=\pm e^t\cdot\cos t$ → K je iz dveh delov: $\vec{r}_1(t)=(e^t\cos t, e^t\sin t, t)$

$\vec{r}_2(t)=(-e^t\cos t, e^t\sin t, t)$

Za nadaljevanje vzemimo $\vec{r}(t)=(e^t\cos t, e^t\sin t, t)$

$$\dot{\vec{r}}(t)=(e^t\cos t-e^t\sin t,\ e^t\sin t+e^t\cos t,\ 1)$$

$$\ddot{\vec{r}}(t)=(\cancel{e^t\cos t}-2e^t\sin t-\cancel{e^t\cos t},\ \cancel{e^t\sin t}+e^t\cos t+e^t\cos t-\cancel{e^t\sin t},\ 0)=(-2\cdot e^t\sin t,\ 2\cdot e^t\cos t,\ 0)$$

$$\dddot{\vec{r}}=(-2e^t\sin t-2e^t\cos t,\ 2e^t\cos t-2e^t\sin t,\ 0)$$

$$\kappa=\frac{|\dot{\vec{r}}\times\ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}\ ;\ \tau=\frac{\dot{\vec{r}}(\ddot{\vec{r}}\times\dddot{\vec{r}})}{|\dot{\vec{r}}\times\ddot{\vec{r}}|^2}$$ → zanimata nas pri $t=0$:

$\dot{\vec{r}}(0)=(1,1,1)\rightarrow|\dot{\vec{r}}|=\sqrt{3}$ $\qquad \dot{\vec{r}}\times\ddot{\vec{r}}=(-2,0,2)$

$\ddot{\vec{r}}(0)=(0,2,0)$

$\dddot{\vec{r}}(0)=(-2,2,0)$ $\qquad \ddot{\vec{r}}\times\dddot{\vec{r}}=(0,0,4)$

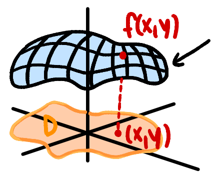
$$\Rightarrow \kappa=\frac{\sqrt{4+4}}{\sqrt{3}^3}=\frac{2}{3}\cdot\sqrt{\frac{2}{3}}\ ;\ \tau(0)=\frac{(1,1,1)\cdot(0,0,4)}{8}=\frac{1}{2}$$

3.12.2020

PLOSKVE

Podajanje ploskev

1) $f: D\subseteq\mathbb{R}^2\longrightarrow\mathbb{R}$ ⇒ ploskev je graf funkcije f:



$\{(x,y,f(x,y));\ (x,y)\in D\}$

EKSPLICITNO PODANA PLOSKEV

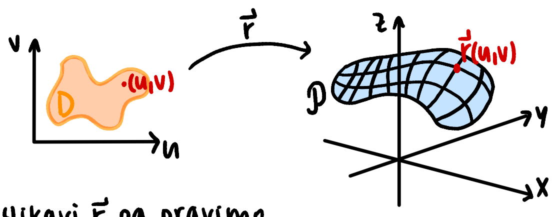
2) IMPLICITNO PODANA PLOSKEV

$$f:\mathbb{R}^3\rightarrow\mathbb{R}\ ;\ \mathcal{P}=\{(x,y,z)\in\mathbb{R}^3;\ f(x,y,z)=0\}$$

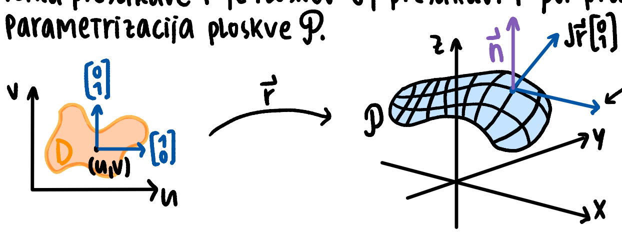
Primer: $f(x,y,z)=x^2+y^2+z^2-1 \Rightarrow \mathcal{P}: x^2+y^2+z^2-1=0 \Leftrightarrow x^2+y^2+z^2=1$ Enotska sfera

3) PARAMETRIČNO PODANA PLOSKEV

$$\vec{r}: D\subseteq\mathbb{R}^2\longrightarrow\mathbb{R}^3$$
$$(u,v)\longrightarrow(x(u,v),y(u,v),z(u,v))$$



Slika preslikave $\vec{r}$ je ploskev $\mathcal{P}$, preslikavi $\vec{r}$ pa pravimo parametrizacija ploskve $\mathcal{P}$.



vektor $\begin{bmatrix}1\\0\end{bmatrix}$ se slika v tangentni vektor, kar pa je ravno odvod preslikave $\vec{r}$:

$J\vec{r}\cdot\begin{bmatrix}1\\0\end{bmatrix}$