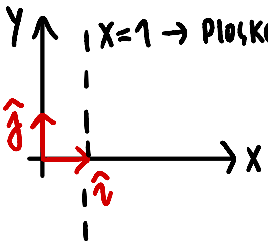
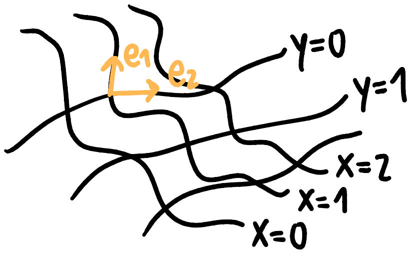
1.VAJE 14.2.2022

Gibanje v krivočrtnih koordinatah

$\vec F = m\cdot\frac{d^2\vec r}{dt^2}$ → En. gibanja zapišemo v določenem koordinatnem sistemu





Poljubne (krivočrtne) koordinate

$$d\vec r = dx\cdot\hat i + dy\cdot\hat j = \frac{\partial\vec r}{\partial x}\cdot dx + \frac{\partial\vec r}{\partial y}\cdot dy$$

V kartezičnih koordinatah

bazni vektor $\vec e_2$

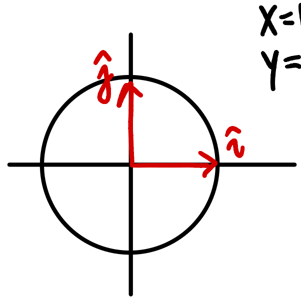
$$d\vec r = \frac{\partial\vec r}{\partial x_1}\cdot dx_1 + \frac{\partial\vec r}{\partial x_2}\cdot dx_2\ , \quad \hat e_1 = \frac{\vec e_1}{|\vec e_1|}$$

bazni vektor $\vec e_1$

↳ tangenten na koordinate

(1) Zapiši kinetično energijo v polarnih koordinatah

$x = r\cdot\cos\varphi$
$y = r\cdot\sin\varphi$



$$T = \frac{1}{2}m\cdot\left(\frac{d\vec r}{dt}\right)^2$$

$$\vec r = x\cdot\hat i + y\cdot\hat j = r\cdot\cos\varphi\cdot\hat i + r\cdot\sin\varphi\cdot\hat j$$

$$d\vec r = \underbrace{dr}_{dx_1}\cdot\cos\varphi\cdot\hat i - r\cdot\sin\varphi\,\underbrace{d\varphi}_{dx_2}\cdot\hat i + \underbrace{dr}_{}\cdot\sin\varphi\,\hat j + r\cdot\cos\varphi\,\underbrace{d\varphi}_{}\,\hat j$$

$$\vec e_1 = \vec e_r = \cos\varphi\cdot\hat i + \sin\varphi\,\hat j = \hat e_r$$
$$\vec e_2 = \vec e_\varphi = r\cos\varphi\,\hat j - r\cdot\sin\varphi\cdot\hat i = r\cdot\hat e_\varphi$$

$$\Rightarrow\ d\vec r = dr\cdot\hat e_r + r\cdot d\varphi\cdot\hat e_\varphi$$

Hitrost: $\vec v = \boxed{\frac{d\vec r}{dt}} = \frac{dr}{dt}\cdot\hat e_r + r\cdot\frac{d\varphi}{dt}\cdot\hat e_\varphi = \boxed{\dot r\hat e_r + r\dot\varphi\hat e_\varphi}$

$$\Rightarrow T = \frac{1}{2}m\cdot\left[\dot r^2\hat e_r^2 + \underbrace{2r\dot r\dot\varphi\,\hat e_r\cdot\hat e_\varphi}_{\emptyset,\ \text{ker } \hat e_r \perp \hat e_\varphi} + r^2\dot\varphi^2\hat e_\varphi^2\right] = \frac{1}{2}m\left[\dot r^2 + r^2\dot\varphi^2\right]$$

D.N.: Izračunaj T v sferičnih koordinatah $\left[T = \frac{1}{2}m\left(\dot r^2 + r^2\sin^2\vartheta\,\dot\varphi^2 + \dot\vartheta^2 r^2\right)\right]$

Sferične koordinate:

$x = r\cdot\cos\varphi\cdot\sin\vartheta$ $\qquad dx_1 = dr$
$y = r\cdot\sin\varphi\cdot\sin\vartheta$ $\qquad dx_2 = d\varphi$
$z = r\cdot\cos\vartheta$ $\qquad dx_3 = d\vartheta$

$$\vec r = x\cdot\hat i + y\cdot\hat j + z\cdot\hat k = r\cdot\cos\varphi\cdot\sin\vartheta\cdot\hat i + r\cdot\sin\varphi\cdot\sin\vartheta\cdot\hat j + r\cdot\cos\vartheta\cdot\hat k$$

$$d\vec{r} = dr\cdot\cos\varphi\cdot\sin\vartheta\cdot\hat{\imath} - r\cdot\sin\varphi\, d\varphi\cdot\sin\vartheta\cdot\hat{\imath} + r\cdot\cos\varphi\cdot\cos\vartheta\cdot d\vartheta\cdot\hat{\imath} +$$
$$+ dr\cdot\sin\varphi\cdot\sin\vartheta\cdot\hat{\jmath} + r\cdot\cos\varphi\, d\varphi\cdot\sin\vartheta\cdot\hat{\jmath} + r\cdot\sin\varphi\cdot\cos\vartheta\, d\vartheta\cdot\hat{\jmath} +$$
$$+ dr\cdot\cos\vartheta\cdot\hat{k} - r\cdot\sin\vartheta\, d\vartheta\,\hat{k}$$

$$\vec{e}_1 = \vec{e}_r = \cos\varphi\cdot\sin\vartheta\cdot\hat{\imath} + \sin\varphi\cdot\sin\vartheta\cdot\hat{\jmath} + \cos\vartheta\cdot\hat{k} = \hat{e}_r$$
$$\vec{e}_2 = \vec{e}_\varphi = -r\cdot\sin\varphi\cdot\sin\vartheta\cdot\hat{\imath} + r\cdot\cos\varphi\cdot\sin\vartheta\cdot\hat{\jmath} = r\cdot\sin\vartheta\cdot\hat{e}_\varphi$$
$$\vec{e}_3 = \vec{e}_\vartheta = r\cdot\cos\varphi\cdot\cos\vartheta\cdot\hat{\imath} + r\cdot\sin\varphi\cdot\cos\vartheta\,\hat{\jmath} - r\cdot\sin\vartheta\cdot\hat{k} = r\cdot\hat{e}_\vartheta$$

$$\to d\vec{r} = dr\cdot\hat{e}_r + d\varphi\cdot r\cdot\sin\vartheta\cdot\hat{e}_\varphi + d\vartheta\cdot r\cdot\hat{e}_\vartheta$$

Hitrost: $\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\cdot\hat{e}_r + \frac{d\varphi}{dt}\cdot r\sin\vartheta\cdot\hat{e}_\varphi + \frac{d\vartheta}{dt}\cdot r\cdot\hat{e}_\vartheta = \dot{r}\hat{e}_r + \dot{\varphi} r\sin\vartheta\hat{e}_\varphi + \dot{\vartheta} r\hat{e}_\vartheta$

$$\to T = \frac{1}{2}m\left[\dot{r}\hat{e}_r + \dot{\varphi} r\sin\vartheta\hat{e}_\varphi + \dot{\vartheta} r\hat{e}_\vartheta\right]\cdot\left[\dot{r}\hat{e}_r + \dot{\varphi} r\sin\vartheta\hat{e}_\varphi + \dot{\vartheta} r\hat{e}_\vartheta\right] =$$

Vsi mešani členi bodo enaki 0, ker $\hat{e}_r \perp \hat{e}_\varphi \perp \hat{e}_\vartheta$

$$= \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\varphi}^2\sin^2\vartheta + r^2\dot{\vartheta}^2\right] \checkmark$$

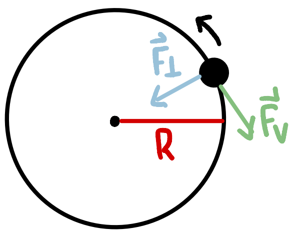
(2) Zapiši 2. Newtonov zakon v polarnih koordinatah

$$\frac{d\vec{r}}{dt} = \dot{r}\cdot\hat{e}_r + r\dot{\varphi}\cdot\hat{e}_\varphi \qquad \hat{e}_r = \hat{\imath}\cdot\cos\varphi + \hat{\jmath}\cdot\sin\varphi$$
$$\frac{d^2\vec{r}}{dt^2} = \ddot{r}\cdot\hat{e}_r + \dot{r}\cdot\dot{\hat{e}}_r + \dot{r}\cdot\dot{\varphi}\cdot\hat{e}_\varphi + r\cdot\ddot{\varphi}\cdot\hat{e}_\varphi + r\dot{\varphi}\cdot\dot{\hat{e}}_\varphi \qquad \hat{e}_\varphi = -\hat{\imath}\cdot\sin\varphi + \hat{\jmath}\cos\varphi$$

$$\boxed{\dot{\hat{e}}_r = -\hat{\imath}\cdot\sin\varphi\frac{d\varphi}{dt} + \hat{\jmath}\cdot\cos\varphi\frac{d\varphi}{dt} = \hat{e}_\varphi\cdot\frac{d\varphi}{dt} = \hat{e}_\varphi\cdot\dot{\varphi}}$$
$$\boxed{\dot{\hat{e}}_\varphi = -\hat{\imath}\cdot\cos\varphi\frac{d\varphi}{dt} - \hat{\jmath}\cdot\sin\varphi\frac{d\varphi}{dt} = -\hat{e}_r\cdot\frac{d\varphi}{dt} = -\hat{e}_r\cdot\dot{\varphi}}$$

$$\Rightarrow \boxed{\frac{d^2\vec{r}}{dt^2}} = \ddot{r}\cdot\hat{e}_r + \dot{r}\cdot\dot{\varphi}\cdot\hat{e}_\varphi + \dot{r}\dot{\varphi}\hat{e}_\varphi + r\ddot{\varphi}\hat{e}_\varphi - r\dot{\varphi}\cdot\dot{\varphi}\,\hat{e}_r =$$
$$= \ddot{r}\cdot\hat{e}_r + 2\cdot\dot{r}\dot{\varphi}\,\hat{e}_\varphi + r\ddot{\varphi}\hat{e}_\varphi - r\cdot(\dot{\varphi})^2\,\hat{e}_r =$$
$$= \boxed{(\ddot{r} - r(\dot{\varphi})^2)\hat{e}_r + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\hat{e}_\varphi}$$

(3) Delec na vodoravno postavljenem obroču



Delec ustavlja viskoznost: $\vec F_v = -\eta \vec v = -\eta \dot{\vec r} = -\eta R\dot\varphi\, \hat e_\varphi$

↳ Ker se delec giblje po obroču je $r = R = \text{konst.}$

$$m\frac{d^2\vec r}{dt^2} = m[-R\dot\varphi^2 \hat e_r + R\ddot\varphi\, \hat e_\varphi] = \underbrace{-F_\perp \cdot \hat e_r - F_v \cdot \hat e_\varphi}_{\text{vse sile, ki delujejo na delec}}$$

$$\varphi:\ m\cancel{R}\ddot\varphi \cdot \hat e_\varphi = -\eta\cancel{R}\dot\varphi\, \hat e_\varphi \Rightarrow m\ddot\varphi = -\eta\dot\varphi \xRightarrow{\omega=\dot\varphi} \dot\omega = -\frac{\eta}{m}\cdot\omega = -\frac{1}{\tau}\cdot\omega$$

$1/\tau = \eta/m$

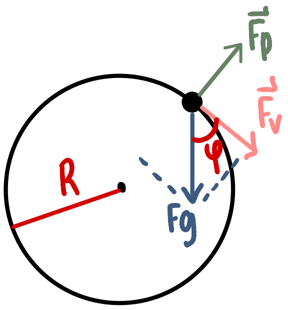
$$\int \Rightarrow \omega = \omega_0 \cdot e^{-t/\tau} = \dot\varphi$$

$$\Rightarrow \varphi = -\underbrace{\omega_0}_{\text{hitrost ob času } t=0}\tau \cdot e^{-t/\tau} + \underbrace{\varphi_0} = -\omega_0\tau e^{-t/\tau} + \omega_0\tau = \omega_0\tau\cdot[1 - e^{-t/\tau}]$$

↳ Kot ob času $t=0$: $\varphi(t=0) = 0 \Rightarrow \varphi_0 = \omega_0\tau$

*S silo vezi (smer $\hat e_r$) se nismo ukvarjali, zato, ker smo izbrali primerne koordinate.

D.N.: Delec z maso m se giblje po navpično postavljenem obroču s polmerom R, po katerem drsi tako, da je gibanje viskozno dušeno (pojemek zaradi viskoznosti je enak $\mathbf{F}_v = -\eta\dot{\mathbf{r}}$). Zapiši Newtonov zakon v polarnih koordinatah. Poišči ravnovesne lege. Ob $t = 0$ delec za majhen kot $\delta\varphi$ odmaknemo iz a) stabilne; b) labilne ravnovesne lege. Zapiši položaj delca ob kasnejših časih!



Newtonov zakon:

$$m\frac{d^2\vec r}{dt^2} = m[-R\dot\varphi^2 \hat e_r + R\ddot\varphi\, \hat e_\varphi] = F_p \cdot \hat e_r - F_v \cdot \hat e_\varphi - F_g\cdot\cos\varphi\cdot \hat e_\varphi - F_g\cdot\sin\varphi\cdot\hat e_r$$

Zanima nas le gibanje v smeri φ:

$$mR\ddot\varphi\, \hat e_\varphi = -F_v \hat e_\varphi - F_g\cdot\cos\varphi\, \hat e_\varphi = -\eta R\dot\varphi\, \hat e_\varphi - mg\cdot\cos\varphi\, \hat e_\varphi$$

$$mR\ddot\varphi = -\eta R\dot\varphi - mg\cdot\cos\varphi \quad /:mR$$

$$\Rightarrow \ddot\varphi + \frac{\eta}{m}\dot\varphi + \frac{g}{R}\cdot\cos\varphi = 0 \ ✓$$

Ravnovesna lega bo tam, kjer bo $\dot\varphi = 0 \Rightarrow mg\cdot\cos\varphi = 0 \Rightarrow \cos\varphi = 0 \Rightarrow \varphi = \frac{\pi}{2}, -\frac{\pi}{2}$

labilna: $\frac{\pi}{2}$, stabilna: $-\frac{\pi}{2}$

a) Ob $t=0$ odmaknemo delec za $\delta\varphi$ iz stabilne lege: $\varphi = -\frac{\pi}{2} + \delta\varphi$

$$\Rightarrow \cos(-\tfrac{\pi}{2} + \delta\varphi) = \cancelto{0}{\cos(-\tfrac{\pi}{2})}\cdot\cos(\delta\varphi) - \underbrace{\sin(-\tfrac{\pi}{2})}_{-1}\cdot\sin(\delta\varphi) = \sin(\delta\varphi) \approx \delta\varphi$$

Enačba se zdaj glasi: $\ddot{\varphi} + \overbrace{\frac{\eta}{m}}^{\beta}\dot{\varphi} + \overbrace{\frac{g}{R}}^{\omega_0^2}\varphi = 0$

↓

Uporabimo nastavek $\varphi = e^{\lambda t}, \; \dot{\varphi} = \lambda e^{\lambda t}, \; \ddot{\varphi} = \lambda^2 e^{\lambda t}$ Kako vem, ali je to > ali < 1 ?

$$\Rightarrow \lambda^2 e^{\lambda t} + \beta \cdot \lambda e^{\lambda t} + \omega_0^2 e^{\lambda t} = 0 \quad /: e^{\lambda t}$$

$$\Rightarrow \lambda^2 + \beta \cdot \lambda + \omega_0^2 = 0, \quad D = \beta^2 - 4\omega_0^2 = \frac{\eta^2}{m^2} - 4 \cdot \frac{g}{R} = \frac{\overset{?}{\eta^2 R - 4mg}}{\underset{>0}{m^2 R}} = \frac{1}{m^2 R}\left[1 - \frac{4mg}{\eta^2 R}\right]$$

$$\lambda_{1,2} = \frac{-\beta \pm \sqrt{D}}{2} = -\frac{\beta}{2} \pm \frac{1}{2}\sqrt{\beta^2 - 4\omega_0^2} = -\frac{1}{2}\left(\beta \mp \sqrt{\beta^2 - 4\omega_0^2}\right)$$

$$\Rightarrow \varphi = e^{\lambda_1 t} + e^{\lambda_2 t} = e^{-\frac{1}{2}\left(\beta + \sqrt{\beta^2 - 4\omega_0^2}\right)t} + e^{-\frac{1}{2}\left(\beta - \sqrt{\beta^2 - 4\omega_0^2}\right)} =$$

$$= e^{-\frac{1}{2}\beta t} \cdot \left[e^{-\frac{1}{2}\sqrt{D}t} + e^{+\frac{1}{2}\sqrt{D}t}\right]$$

Če je $\sqrt{D} \in \mathbb{C} \Rightarrow i\sqrt{\tilde{D}}, \; \sqrt{\tilde{D}} \in \mathbb{R} \Rightarrow \varphi = 2 \cdot e^{-\frac{1}{2}\beta t} \cdot \frac{e^{-\frac{1}{2}i\sqrt{\tilde{D}}t} + e^{+\frac{1}{2}i\sqrt{\tilde{D}}t}}{2} =$

$$\varphi = 2e^{-\frac{1}{2}\beta t} \cdot \cos\left(\frac{1}{2}\sqrt{4\omega_0^2 - \beta^2} \cdot t\right) \; ?$$

(4) Gibanje prostega delca v polarnih koordinatah

↳ v kartezičnih: $\ddot{\vec{r}} = 0 \Rightarrow \dot{\vec{r}} =$ konst $\Rightarrow$ gibanje je enakomerno v vseh treh smereh

$$\vec{F} = m \cdot \vec{a} = 0 \Rightarrow \frac{d^2\vec{r}}{dt^2} = \underbrace{(\ddot{r} - \underbrace{r(\dot{\varphi})^2}_{r\omega^2})}_{\emptyset}\hat{e}_r + \underbrace{(\underbrace{2\dot{r}\dot{\varphi}}_{\text{Coriolis}} + r\ddot{\varphi})}_{\emptyset}\hat{e}_\varphi = 0$$

(1) $\ddot{r} - r \cdot \dot{\varphi}^2 = 0$

(2) $2\dot{r}\dot{\varphi} + r \cdot \ddot{\varphi} = 0 \quad / \cdot r$

$\lambda =$ konst. → konstanta gibanja

$$2r\dot{r}\dot{\varphi} + r^2\ddot{\varphi} = 0 \rightsquigarrow \frac{d}{dt}(r^2\dot{\varphi}) = 0 \longrightarrow \dot{\varphi}^2 = \frac{\lambda^2}{r^4}$$

→ (1) $\ddot{r} - \frac{\lambda^2}{r^3} = 0 \quad / \cdot \dot{r} \Rightarrow 2 \cdot \left[\dot{r} \cdot \ddot{r} - \frac{\lambda^2}{r^3}\dot{r}\right] = 0 = \left(\dot{r}^2 + \frac{\lambda^2}{r^2}\right)^{\cdot} \Rightarrow \dot{r}^2 + \frac{\lambda^2}{r^2} = \varepsilon =$ konst.

$r(t) = r(\varphi(t)) \rightarrow$ s tem nastavkom gremo v enačbo za ε:

$$\dot{r} = \frac{dr}{d\varphi} \cdot \dot{\varphi} = \frac{dr}{d\varphi} \cdot \frac{\lambda}{r^2}$$

$u = 1/r$
$r = 1/u$
$r' = -1/u^2 \cdot u' = -r^2 u'$

$$\Rightarrow \left(\frac{dr}{d\varphi} \cdot \frac{\lambda}{r^2}\right)^2 + \frac{\lambda^2}{r^2} = r'^2 \cdot \frac{\lambda^2}{r^4} + \frac{\lambda^2}{r^2} = \varepsilon \longrightarrow \cancel{r^4} \cdot u'^2 \cdot \frac{\lambda^2}{\cancel{r^4}} + \lambda^2 u^2 = \varepsilon$$

$$\Rightarrow \underbrace{\lambda^2}_{\text{konst.}}(u'^2 + u^2) = \underbrace{\varepsilon}_{\text{konst}} \Rightarrow u = A \cdot \cos(\varphi - \varphi_0) \Rightarrow r = \frac{r_0}{\cos(\varphi - \varphi_0)}$$

⇓

$u'^2 + u^2 =$ konst.